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Lie–Bäcklund groups and the linearisation of differential equations

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Abstract. It is shown that groups of Lie–Bäcklund (LB) transformations which depend on non-local variables are related by a change of variables to the LB tangent transformations of Ibragimov and Anderson, involving no more than arbitrary-order derivatives. The transformation of any LB symmetry operator by an invertible change of variables is discussed. It is pointed out that once a differential equation admits an LB operator, then a large number of 'secondary' equations will admit the same operator. The LB theory involving non-local variables and the notion of secondary equations are used to characterise group theoretically the linearisation of the Burgers equation, $u_t + uu_x - u_{xx} = 0$, and of the ODE $u_{xx} + \omega^2(x)u + Ku^{-3} = 0$.

1. Introduction

The earliest attempts at generalisation of the first-order tangent (contact) transformations of Lie (1874a) were made by Lie himself (Lie 1874b, 1880) and by Bäcklund (1873, 1876, 1880, 1882) but it is only in recent times that the precise group-theoretical context of these generalisations was established in the work of Ibragimov and Anderson (1977) and Anderson and Ibragimov (1979). The last reference contains a concise historical account of the work of Lie, Bäcklund and others followed by an account of the authors' own contributions to the field of groups of continuous transformations. This consists of an elaboration of the theory of the Lie–Bäcklund (LB) groups of transformations in which derivatives of arbitrary order appear (infinite-order tangent-transformation groups). More recently still, Konopelchenko and Mokhnachev (1979, 1980) have augmented the theory of these groups to include transformations which depend on integrals ('non-local' variables).

Section 2 of the present paper begins with a short account of the theory of the LB groups of Anderson and Ibragimov. The results of Konopelchenko and Mokhnachev are then reobtained in a way which shows that the groups of transformations introduced by these authors are related to the LB transformations of Ibragimov and Anderson (1977) by a change of variables. This probably means that many developments in the theory of LB groups depending on derivatives can be carried over directly to these new kinds of groups. It appears that the starting point of such considerations is the work of Pirani *et al* (1979) and of Kosmann-Schwarzbach (1979). The application of LB groups to the study of differential, integrodifferential and integral equations is also discussed in § 2 and an example is given involving the generator of an LB group depending on an integral.

A situation frequently found in applications is that a given differential equation is transformed into a new one under the action of a transformation (change of variables) which in general is *not* an LB transformation. Section 3 deals with the corresponding transformation of operators acting on the original equation into operators acting on the new equation.

Section 4 begins with an account of certain groups of LB transformations which are useful in the study of *nonlinear* differential equations and which may be related to the linearisability of those equations. In general these transformations will depend on integrals of functions. In this section they are applied to the study of a number of specific ordinary and partial differential equations which are known, *ab initio*, to be linearisable. The particular examples were chosen for the variety of different aspects of this kind of application which they exhibit and some of them are worked out in detail. It is found possible to characterise group theoretically the linearisation of each equation considered.

The precise group-theoretical characterisations set down here are new results and they hinge largely on the ability to handle groups of transformations involving integrals within the LB formalism.

2. Generalisation of the infinitesimal generators of groups of Lie–Bäcklund transformations

In the notation of Anderson and Ibragimov (1979) the infinitesimal generator (LB symmetry operator) of a LB group G of transformations is

$$\hat{X}=\xi^i\frac{\partial}{\partial x^i}+\eta^\alpha\frac{\partial}{\partial u^\alpha}+\zeta_i^\alpha\frac{\partial}{\partial u_i^\alpha}+\ldots+\zeta^\alpha_{i_1\ldots i_s}\frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}} \tag{2.1}$$

where $s=1,2,3,\ldots$ and summation is implied over repeated indices. If a is the group parameter then the 'coordinates' ξ^i, η^α, etc, give the first-order terms in the power series expansions in a of the variables x^i, u^α, etc. Thus for example

$$x'^1=x^1+\xi^1a+\ldots.$$

The group G of transformations is a group of LB transformations iff the coordinates of (2.1) satisfy the recurrence relations

$$\zeta_i^\alpha=D_i(\eta^\alpha)-u_j^\alpha(D_i(\xi^j)\qquad \zeta^\alpha_{i_1i_2}=D_{i_2}(\zeta^\alpha_{i_1})-u^\alpha_{i_1j}D_{i_2}(\xi^j)\qquad\ldots \tag{2.2}$$

where D_i is the operator of 'total differentiation' with respect to the variable x_i, that is

$$D_i=\frac{\partial}{\partial x_i}+u_i^\alpha\frac{\partial}{\partial u^\alpha}+u^\alpha_{ii_1}\frac{\partial}{\partial u^\alpha_{i_1}}+u^\alpha_{ii_1i_2}\frac{\partial}{\partial u^\alpha_{i_1i_2}}+\ldots \tag{2.3}$$

which acts on functions of the independent variables x^i, u^α, $\underset{1}{u}^\alpha$, $\underset{2}{u}^\alpha$, ... Moreover each generator of the type (2.1) has an equivalent simpler form (Anderson and Ibragimov 1979, p 63) in which all the ξ^i are zero. This equivalence concept and relations (2.2) mean that only operators of the form

$$\hat{X}=\eta^\alpha\frac{\partial}{\partial u^\alpha}+(D_i\eta^\alpha)\frac{\partial}{\partial u_i^\alpha}+(D_{i_1}D_{i_2}\eta^\alpha)\frac{\partial}{\partial u^\alpha_{i_1i_2}}+\ldots \tag{2.4}$$

need be considered. As soon as the first (or 'defining') term in $\hat{X}$ is known the rest follows by 'prolongation' or 'extension', via (2.2). For this reason an operator is often designated by its defining term. The defining term of the equivalent form of (2.1) is given by

$$\hat{X}_{\text{eq}} = (\eta^\alpha - \xi^i u_i^\alpha)\, \partial/\partial u^\alpha \equiv \tilde{\eta}^\alpha\, \partial/\partial u^\alpha. \tag{2.5}$$

In a particular application only part of $\hat{X}$ is needed and this is called the 'relevant' part.

Consider now a given system of differential equations

$$\omega : \omega_\nu\left(x, u, \underset{1}{u}, \ldots, \underset{k}{u}\right) = 0 \qquad \nu = 1, \ldots, M \tag{2.6}$$

and suppose that the operator $\hat{X}$ in (2.4) generates a LB transformation group G.

Definition 2.1. The system of differential equations (2.6) is invariant with respect to the Lie–Bäcklund group G, if the manifold given by the following infinite system of differential equations

$$\omega_\nu = 0 \qquad D_i\omega_\nu = 0 \qquad D_iD_j\omega_\nu = 0 \qquad \ldots \qquad \nu = 1, \ldots, M \tag{2.7}$$

is an invariant manifold of the group G.

Theorem 2.1 (Anderson and Ibragimov 1979). The system of differential equations (2.6) is invariant with respect to a Lie–Bäcklund transformation group G generated by a Lie–Bäcklund operator of the form (2.4) iff

$$\hat{X}\omega_\nu|_{\omega_\nu \doteq 0} = 0 \qquad \nu = 1, \ldots, M \tag{2.8}$$

where $\omega_\nu \doteq 0$ is notation for the manifold defined by equations (2.7). In this case the set ω is said to 'admit' the operator $\hat{X}$.

2.1. *Introduction of non-local variables into the theory of Lie-Bäcklund groups*

Operators of the type so far considered cannot be used as they stand in connection with integral equations or integrodifferential equations, since they do not involve integrals of functions of the independent variables $(x^i, u^\alpha, \underset{1}{u}^\alpha, \underset{2}{u}^\alpha, \ldots)$. Also one can find differential equations which admit operators depending on integrals (non-local symmetry generators) in, for example, Fushchich (1971, 1974) and in Lüscher and Pohlmeyer (1978). Konopelchenko and Mokhnachev (1979, 1980) were the first to show how to augment LB group techniques to take account of integrals. These authors' results will now be reobtained and amplified by a somewhat different approach, which shows that the operators introduced by them are formally related to the LB operators of Anderson and Ibragimov by a change of variables. To this end consider the operator (2.4) for the special case of only one u and one x,

$$\hat{X} = \eta\, \partial/\partial u + D\eta\, \partial/\partial(Du) + DD\eta\, \partial/\partial(DDu) + \ldots, \tag{2.9}$$

where $Du = u_x$ has been employed and introduce the change of variables

$$Du = v \qquad \text{with } u = D^{-1}v. \tag{2.10}$$

Here D^{-1} is the inverse of the operator of total differentiation and $D^{-1}v$ is to be considered a new v variable, functionally independent of and on the same footing as v, v_x, etc. In a specific problem it may be possible to give an explicit form for the

D^{-1} operator although for the most part in the present work it will be used formally. If v vanishes sufficiently rapidly as $x \to -\infty$ we can self-consistently take $D^{-1}v = \int_{-\infty}^{x} v(x')\,dx'$. The operator D^{-1} is often referred to as a non-local operator and the variable $D^{-1}v$ as a non-local variable. Note that $D^{-1}v_x = v$, $D^{-1}v_{xx} = v_x$, etc. Some previous examples of the use of the D^{-1} operator and reference to some of its properties may be found in Kruskal *et al* (1970) and in Olver (1977).

The transformation (2.10) is invertible in the sense that all the u variables may be expressed in terms of the v variables and *vice versa*. This is necessary in order to transform the operator (2.9) into the operator

$$\begin{aligned}\tilde{X} &= \eta\,\partial/\partial(D^{-1}v) + D\eta\,\partial/\partial v + DD\eta\,\partial/\partial(Dv) + \ldots \\ &= D^{-1}\tilde{\eta}\,\partial/\partial(D^{-1}v) + \tilde{\eta}\,\partial/\partial v + D\tilde{\eta}\,\partial/\partial(Dv) + \ldots\end{aligned} \tag{2.11}$$

having written $\tilde{\eta} = D\eta$.

Evidently the process by which (2.11) was obtained may be repeated as often as desired or (2.10) may be replaced with

$$D^n u = v \qquad u = D^{-n} v$$

thereby introducing into (2.11) the partial derivatives with respect to $D^{-n}v$ for arbitrary n. The extension to more than one u and more than one x is obvious. Note that many new independent variables are introduced in this process. The resulting operator takes the form

$$\begin{aligned}\hat{X} = \ldots &+ (D_{i_1}^{-1}D_{i_2}^{-1}\ldots D_{i_r}^{-1}\eta^\alpha)\,\partial/\partial(D_{i_1}^{-1}D_{i_2}^{-1}\ldots D_{i_r}^{-1}u^\alpha) + \ldots + \eta^\alpha\,\partial/\partial u^\alpha + \ldots \\ &+ D_{i_1}D_{i_2}\ldots D_{i_s}\eta^\alpha\,\partial/\partial(D_{i_1}D_{i_2}\ldots D_{i_s}u^\alpha) + \ldots.\end{aligned} \tag{2.12}$$

With appropriate restrictions on the u^α, operators of the type (2.12) are useful in studying the invariance of integral and integrodifferential equations involving simple integrals of the type $D_{i_1}^{-1}D_{i_2}^{-1}\ldots D_{i_r}^{-1}u^\alpha$. Any operator of the form (2.12) is a Lie–Bäcklund operator in the infinite-dimensional space of independent variables $(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, D^{-1}u, D^{-2}u, \ldots)$ where the $x, u, \underset{1}{u}$ are as before and $D^{-r}u$ is the set of non-local variables $D_{i_1}^{-1}D_{i_2}^{-1}\ldots D_{i_r}^{-1}u^\alpha$ $(\alpha = 1, \ldots, m;\ i_r = 1, \ldots, n)$ because it can be obtained from (2.4) by a change of variables.

With the introduction of a set of new independent variables it is necessary to extend the definition of D_i in (2.3) to

$$\begin{aligned}D_i = \frac{\partial}{\partial x^i} + u_i^\alpha\frac{\partial}{\partial u^\alpha} + u_{ii_1}^\alpha\frac{\partial}{\partial u_{i_1}^\alpha} + u_{ii_1i_2}^\alpha\frac{\partial}{\partial u_{i_1i_2}^\alpha} + \ldots + \frac{\partial(D_{i_1}^{-1}u^\alpha)}{\partial x^i}\,\frac{\partial}{\partial(D_{i_1}^{-1}u^\alpha)} \\ + \frac{\partial(D_{i_1}^{-1}D_{i_2}^{-1}u^\alpha)}{\partial x^i}\,\frac{\partial}{\partial(D_{i_1}^{-1}D_{i_2}^{-1}u^\alpha)} + \ldots.\end{aligned}$$

The form of the LB operator necessary to consider the invariance of equations involving *integrals* of all the independent variables (some of which are themselves integrals) so far introduced into the formalism will be obtained next. In other words, letting (u) be notation for dependence on any subset of the variables $(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, D^{-1}u, D^{-2}u, \ldots)$, we shall obtain the form of the operator acting on integrals of functions, of the form $D^{-r}f(u)$. First note that, using the integral notation of Konopelchenko and Mokhnachev (1979, 1980), (2.12) may be written in the rather different form

$$\hat{X} = \int \mathrm{d}f\Big(\ldots + (D_{i_1}^{-1} D_{i_2}^{-1} \ldots D_{i_r}^{-1} \eta^\alpha) \frac{\partial f(u)}{\partial (D_{i_1}^{-1} D_{i_2}^{-1} \ldots D_{i_r}^{-1} u^\alpha)} + \ldots + \eta^\alpha \frac{\partial f(u)}{\partial u^\alpha} + \ldots$$

$$+ D_{i_1} D_{i_2} \ldots D_{i_s} \eta^\alpha \frac{\partial f(u)}{\partial (D_{i_1} D_{i_2} \ldots D_{i_s} u^\alpha)} + \ldots \Big) \frac{\partial}{\partial f(u)} \tag{2.13}$$

which may be written

$$\hat{X} = \int \mathrm{d}f \, \eta_f(u) \, \partial/\partial f(u) \tag{2.14}$$

where η_f is the quantity in brackets. The notation $\int \mathrm{d}f$ means a sum over distinct functions f which are independent in the sense that $\partial h/\partial f = \delta(h - f)$.

By regarding $f(u)$ as some new variable, w, each term in (2.14) can be extended to integrals of $f(u)$ thus introducing further new variables into the formalism (extension to derivatives of $f(u)$ is redundant in that each such derivative is some new function $\tilde{f}(u)$). Unfortunately this last extension introduces an element of multiple counting; for example $D^{-r}u$ appears twice. The important point, however, is that if $\{Z\}$ is the set of all independent u variables, then by iteration of the process so far described the coordinate η_Z corresponding to any independent non-local variable Z can be calculated. If (2.13) and (2.14) are reinterpreted so that (u) denotes dependence on all possible independent variables and η_f includes the partial derivatives with respect to all such variables, then (2.13) and (2.14) define the most general Lie–Bäcklund operator. Therefore we have established the result that the most general LB operator is formally related to the LB operators of Anderson and Ibragimov by a change of variables. This suggests that many of the results obtained to date in the theory of LB operators will carry over to the more general type of operator considered here. It would be of interest, for example, to see to what extent the results in Kosmann-Schwarzbach (1979) and in Pirani *et al* (1979) carry over to the present situation. This work is based on the jet-bundle formalism of Ehresman (1953). It is our intention to investigate whether this formalism generalises to include non-local variables.

Having established the desired formal relationship mentioned above it should be noted that η_Z is most easily calculated by the infinitesimal method as employed by Konopelchenko and Mokhnachev (1979, 1980). To do this, one simply finds the coefficient of the first power of the parameter a in the power series expansion Z. For example if $Z = D_{x^1}^{-1} u D_{x^2}^{-1} u D_{x^3}^{-1} u$ then η_Z is given by

$$\eta_Z = D_{x^1}^{-1} \eta D_{x^2}^{-1} u D_{x^3}^{-1} u + D_{x^1}^{-1} u D_{x^2}^{-1} \eta D_{x^3}^{-1} u + D_{x^1}^{-1} u D_{x^2}^{-1} u D_{x^3}^{-1} \eta.$$

The final generalisation of the total differential operator D_i is given by

$$D_i = \sum_{\{Z\}} (\partial Z/\partial x^i)(\partial/\partial Z) \tag{2.15}$$

which acts on functions of the set of independent variables $\{Z\}$. An example of the use of the extended D_i operator will be given in § 4. Refer to equations (4.11) and the remarks that follow.

2.2. Application to equations of Lie–Bäcklund groups involving non-local variables

Definition (2.1) and theorem (2.1) require modification following the introduction of non-local variables. In the notation introduced in the previous subsection, consider a system of differential, integrodifferential or integral equations

$$\omega : \omega_\nu(u) = 0 \qquad \nu = 1, \ldots, M \tag{2.16}$$

and suppose the operator $\hat{X}$ in (2.14) generates a LB group G.

Definition 2.2. The system (2.16) is invariant with respect to the Lie–Bäcklund group G if the manifold given by the system of equations, consisting of (2.16) and its general integrodifferential consequences,

$$\begin{array}{llll} \ldots & D_i^{-1}D_j^{-1}\omega_\nu = 0 & D_i^{-1}\omega_\nu = 0 & \omega_\nu = 0 \\ D_i\omega_\nu = 0 & D_iD_j\omega_\nu = 0 & \ldots & \end{array} \tag{2.17}$$

forms an invariant manifold of the group G with $\nu = 1, \ldots, M$.

By following the method of Anderson and Ibragimov (1979, pp 62–3) and using the identities (Ibragimov 1976, Konopelchenko and Mokhnachev 1979, 1980)

$$\hat{X}D_i - D_i\hat{X} = 0 \tag{2.18a}$$

$$\hat{X}D_i^{-1} - D_i^{-1}\hat{X} = 0 \tag{2.18b}$$

it is easy to obtain the following modifications of theorem 2.1.

Theorem 2.2. The system of differential equations (2.16) is invariant with respect to a Lie–Bäcklund group generated by a Lie–Bäcklund operator of the form (2.14) iff

$$\hat{X}\omega_\nu|_{\omega_\nu \doteq 0} = 0 \qquad \nu = 1, \ldots, M \tag{2.19}$$

where $\omega_\nu \doteq 0$ stands for the manifold defined by equations (2.17).

It is obvious that each set of M equations in (2.17) is on the same footing as all the other such sets and that a LB operator $\hat{X}$ admitted by one set of equations is admitted by all its differential and integral consequences. These results were first noted by Ibragimov (1976) for differential consequences, and by Konopelchenko and Mokhnachev (1979, 1980) for integral consequences. However, the invariance of integral consequences under groups of LB transformations is a generalisation of a well known result in the theory of Lie point groups. Suppose a single (for illustration purposes) differential equation, $K(u) = 0$, is invariant under a Lie point group, G, of transformations. An equivalent statement is that the solution manifold of $K(u) = 0$ is transformed into itself under G. Accordingly, so long as all possible initial values are allowed, the first integral of $K(u) = 0$, $\int_{x_0}^{x} K(u) = 0$, is also invariant under the group G. On the other hand, if a particular initial condition (constant of integration) is chosen, then the resulting differential equation has a lesser invariance group of transformations than does equation $K(u) = 0$.

2.3. Secondary equations

Given that a system of differential, integrodifferential or integral equations admits a certain LB operator $\hat{X}$, then many related systems will exist (beyond differential and integral consequences) which admit the same operator. To see how this can occur we prove a simple proposition for a single equation.

Proposition 2.1. Let

$$\hat{X}\omega(u)|_{\omega(u) \doteq 0} = 0 \tag{2.20}$$

and

$$\hat{X}f(u)=g(u)f(u) \tag{2.21}$$

for two functions g and f where, as previously, (u) denotes dependence on one or more of the u variables; then

$$\hat{X}[f(u)\omega(u)]|_{f(u)\omega(u)\doteq 0}=0, \tag{2.22}$$

that is, the 'secondary equation' $f(u)\omega(u)=0$ also admits $\hat{X}$.

Proof. The left-hand side of equation (2.22) yields

$$[g(u)f(u)\omega(u)+f(u)\hat{X}\omega(u)]|_{f(u)\omega(u)\doteq 0}. \tag{2.23}$$

When $f(u)\omega(u)\doteq 0$ then either $f(u)=0$ or $\omega(u)=0$ (or both equal zero). When $f(u)=0$ (2.23) vanishes trivially. When $\omega(u)=0$ it also vanishes using (2.20), and so we have the result (2.22).

The converse of this proposition is also true. To see this, first note that

$$\hat{X}f^{-1}(u)=-f^{-2}(u)\hat{X}f(u)=-g(u)f^{-1}(u)$$

and then apply this result to $f(u)\omega(u)=0$ to obtain (2.20).

The above is just one of many ways in which secondary equations can be constructed beginning with some basic equation $\omega(u)=0$. Some concrete examples appear in § 4.

Bearing in mind that an operator admitted by the secondary equation is also admitted by its consequences, it becomes clear that the operator of a LB group of transformations may be admissible by a very large number of differential equations. This fact assumes some importance in relation to linearisation of equations, as discussed in § 4.

2.4. *Example of the use of non-local variables*

Consider the well known Burgers equation, a model for turbulence,

$$u_t+uu_x-u_{xx}=0. \tag{2.24}$$

If, for all t, the function u vanishes sufficiently rapidly as $x\to-\infty$ then D_x^{-1} may be interpreted as $\int_{-\infty}^{x}\mathrm{d}x'$, in other words there is never any contribution at the lower limit of integration. Then on integration equation (2.24) becomes

$$\omega=D_x^{-1}u_t+\tfrac{1}{2}u^2-u_x=0. \tag{2.25}$$

Equation (2.25) admits the LB operator

$$\hat{Y}=(2h_x+hu)\exp(\tfrac{1}{2}D_x^{-1}u)\,\partial/\partial u+\dots \tag{2.26}$$

with the relevant part

$$\begin{aligned}\hat{Y}_{\mathrm{rel}}=\hat{Y}&+(2h_t+hD_x^{-1}D_tu)\exp(\tfrac{1}{2}D_x^{-1}u)\,\partial/\partial(D_x^{-1}D_tu)\\&+(2h_{xx}+2h_xu+hu_x+\tfrac{1}{2}hu^2)\exp(\tfrac{1}{2}D_x^{-1}u)\,\partial/\partial(D_xu)\end{aligned} \tag{2.27}$$

where $h(t,x)$ is an arbitrary solution of the linear heat equation, that is $h_t-h_{xx}=0$.

The proof is as follows. Employing (2.27) we have

$$\hat{Y}\omega|_{\omega \doteq 0} = (2h_t + hD_x^{-1}D_t u + hu^2 - 2h_{xx} - hu_x - \tfrac{1}{2}hu^2)\exp(\tfrac{1}{2}D_x^{-1}u)|_{\omega \doteq 0} = 0$$

using (2.25) only. The Burgers equation (2.24), as a differential consequence of (2.25), also admits the operator (2.26). See Ibragimov (1980) where this is discussed without the use of non-local variables, but at the expense of introducing an extra equation, the so called Burgers potential equation.

3. Transformation of Lie–Bäcklund operators

Since many differential equations, especially nonlinear ones, are intractable in their original form, much effort has been invested in finding transformations to new equations which can be solved more easily. These transformations are rarely (if ever) groups of LB transformations. Indeed care must be taken to distinguish between a LB group of transformations (G) under which a given equation is invariant and a transformation (T) under which the given equation is transformed into a new equation. In general T will have neither an infinitesimal generator nor the previously discussed property of equivalence. It is of interest to investigate how the generator ($\hat{X}$) of G, under which a given equation is invariant, is transformed when the equation itself is transformed under T. Aspects of this subject have been discussed by Bluman (1974), Peterson (1976), Anderson and Ibragimov (1979) and Ibragimov (1980).

Consider then a given differential, integrodifferential or integral equation, $K(u) = 0$, where here (u) denotes dependence on a finite subset of $\{Z\}$, the set of all independent 'u variables'. Let T be the transformation

$$T: v \to u = u(v) \tag{3.1a}$$

which is shorthand for the fact that under T all of the $\{Z\}$ can be expressed in terms of the $\{\Omega\}$ which is the set of all independent v variables. T is assumed to have an inverse such that

$$T^{-1}: u \to v = v(u). \tag{3.1b}$$

The assumption of an inverse is necessary in order to effect the transformation of a differential operator involving the independent variables (however, a differential equation may be successfully transformed without such invertibility as for example the well known Miura transformation of the Korteweg–de Vries equation which is not known to be invertible in the sense discussed here). With an eye toward the type of application we have in mind a further assumption is made here, namely that (3.1) does not depend explicitly on the (x^i) (see the discussion at the start of § 4).

Now T as applied to $K(u)$ means $K[u(v)]$, that is, a change of variables by substitution and T^{-1} as applied to say $M(v)$ is $M[v(u)]$. Therefore, by the chain rule, for each independent variable Z

$$T\,\partial/\partial Z = \sum_{\{\Omega\}} (\partial\Omega/\partial Z)\,\partial/\partial\Omega$$

and an operator $\hat{X}(u)$ can now be transformed into an operator $\hat{Y}(v)$ and *vice versa.*

Thus

$$T\hat{X}(u)=T\int \mathrm{d}g\,\eta_g(u)\frac{\partial}{\partial g(u)}$$

$$=T\int \mathrm{d}g\left(\sum_{\{Z\}}\eta_Z\frac{\partial g}{\partial Z}\right)\frac{\partial}{\partial g}$$

$$=\int \mathrm{d}f\left[\sum_{\{\Omega\}}\sum_{\{Z\}}\left(\eta_Z\frac{\partial \Omega}{\partial Z}\right)_{u=u(v)}\frac{\partial f}{\partial \Omega}\right]\frac{\partial}{\partial f} \tag{3.2}$$

where $f(v)=g[u(v)]$. This last equation has the useful form

$$T\hat{X}(u)=\int \mathrm{d}f\left(\sum_{\{\Omega\}}(\hat{X}\Omega)_{u=u(v)}\frac{\partial f}{\partial \Omega}\right)\frac{\partial}{\partial f}\qquad i=1,\ldots,n \tag{3.3}$$

$$=\int \mathrm{d}f\,\eta_f\frac{\partial}{\partial f}\equiv \hat{Y}(v). \tag{3.4}$$

Furthermore, by the identity (2.18*a*) we see that (3.3) has the recurrence property (2.2) and $\hat{Y}(v)$ is a Lie–Bäcklund operator. In other words this result verifies that the prolongation structure applies in the new operator.

The above result is also of considerable calculational significance. When using equation (3.2) to transform a given operator it is necessary to find only a single term of the new operator, since the remainder are determined by prolongation and may be found by recurrence for simple terms and by the infinitesimal method used by Konopelchenko and Mokhnachev (1979, 1980) for more complicated terms. Usually the defining term $\eta(v)\,\partial/\partial v$ in the new operator will be sought and so it is necessary to find all the terms in the old operator which contribute to this term in the new one. Just which terms are needed will depend on the variables on the right-hand side of equation (3.1*b*). Some examples appear in § 4.

Now suppose $\hat{X}$ is admitted by $K(u)=0$, that is

$$\hat{X}(u)K(u)|_{K(u)\doteq 0}=0 \tag{3.5}$$

and let this equation be transformed under T to give

$$\hat{Y}(v)M(v)|_{M(v)\doteq 0}=0. \tag{3.6}$$

Therefore, the new equation $M(v)=0$ admits the operator $\hat{Y}(v)$. But if (3.6) admits $\hat{Y}(v)$, then by applying T^{-1} it follows that $K(u)=0$ admits $\hat{X}(u)$ and so the following proposition holds.

Proposition 3.1. The equation (3.6) admits $\hat{Y}(v)$ if and only if the equation $K(u)=0$ admits $\hat{X}(u)$.

The special case when T is a Lie contact transformation has recently been treated by Kumei and Bluman (1982). By hypothesis each Lie contact transformation has an inverse. However, many other possibilities can now be handled. For example the Burgers equation (2.24) is transformed into the equation

$$D_x(\varphi_t+\tfrac{1}{2}\varphi_x^2-\varphi_{xx})=0 \tag{3.7}$$

under

$$u=\varphi_x\qquad \varphi=D_x^{-1}u. \tag{3.8}$$

Equation (3.7) is usually integrated to give the 'potential' equation

$$\varphi_t+\tfrac{1}{2}\varphi_x^2-\varphi_{xx}=0. \tag{3.9}$$

The operator (2.26) admitted by the Burgers equation is readily transformed under (3.8) to obtain the operator (Ibragimov 1980) $h(t,x)\,\mathrm{e}^{\varphi/2}\,\partial/\partial\varphi$, which is admitted by (3.7) and its integral consequence (3.9).

4. Application to the linearisation of differential equations

It is well known that every linear differential equation, $L(v)=0$, admits the operator $h(x^i)\,\partial/\partial v$, where $h(x^i)$ is any solution of $L(v)=0$. If $L(v)$ contains only constant coefficients the linear equation also admits $h_x\,\partial/\partial v$, $h_{xx}\,\partial/\partial v,\ldots$. Some linear differential equations admit more complex LB operators as for example the ordinary differential equation

$$v_{xxx}+4\omega\omega_x v+4\omega^2 v_x=0 \qquad \omega=\omega(x) \tag{4.1}$$

which admits the operator $(h_x v-hv_x)\,\partial/\partial v$, where $h(x)$ is any solution of (4.1). Another situation that can occur is that of the example in § 2.4. In that case the LB operator (2.26) depending on h, an arbitrary solution of the *linear* heat equation $v_x-v_{xx}=0$, is admitted by the *nonlinear* equations (2.24) and (2.25).

Operators such as the above, which depend on an arbitrary solution of some differential equation, will be referred to here as 'free' group operators, to be denoted by the form $\eta(h;v)\,\partial/\partial v$, indicating that h may depend on $(\ldots D_x^{-1}h, h, h_x,\ldots, D_x^{-1}v, v, v_x,\ldots)$. In general, such an operator represents an infinite number of independent operators, one for each solution, h, of the differential equation, but of course when h satisfies an ordinary differential equation the number of independent operators is finite. It follows from the discussion in § 2.3 that many equations related to a given equation, $M(v)=0$, may also admit an operator admitted by $M(v)=0$ itself. For example v times equation (4.1) and also the integral of the resulting equation

$$v_x^2-2vv_{xx}-4\omega^2v^2+K=0 \tag{4.2}$$

both admit the operator $(h_x v-hv_x)\,\partial/\partial v$ admitted by (4.1).

In a recent paper Ibragimov (1980) has drawn attention to the fact that a number of nonlinear differential equations of physical interest admit free operators of the general type $\eta(h;u)\,\partial/\partial u$ involving an arbitrary solution of a linear equation. He uses this fact as the basis for a brief discussion of the group-theoretical background to the linearisability of differential equations and he also presents some examples. More recently Kumei and Bluman (1981) have given necessary and sufficient conditions, in terms of an admissible free group, for a system of nonlinear differential equations, to be linearisable by a Lie tangent transformation (they call it a Lie contact transformation). Since a Lie tangent transformation may depend on no more than first-order derivatives the applicability of the results of Kumei and Bluman (1981) is limited. Our main purpose here is to show that the apparatus developed in the previous sections can be used to consider much more general linearising transformations. We shall obtain a complete group-theoretical characterisation of the known linearisability, by transformations other than Lie tangent transformations, of certain differential equations arising in physical problems.

A basic question to be considered is that if a given nonlinear differential equation, $N(u)=0$, admits a LB operator of the form $\eta(h;u)\,\partial/\partial u$, does that mean it can be 'transformed' into the linear differential equation of which h is a solution? We do not offer a general answer to this question, but based on our work to this point, we expect that any 'transformation' will occur in general in two stages. Firstly, we expect that a transformation T, such as that of § 3, applied to $N(u)=0$ will yield a secondary equation (or its integrodifferential consequence) of the 'target' linear equation. Symbolically

$$TN(u)=\mathscr{F}(v). \tag{4.3}$$

The quantity $\mathscr{F}(v)$ in general differs from $L(v)$ where $L(v)=0$ is the target linear equation. The second stage will consist of the relationship between $\mathscr{F}(v)=0$ and $L(v)=0$. In the first stage the admissible operator is transformed through T and T^{-1} but in the second stage we expect that the admissible operator will remain unaltered. One desirable possibility that can occur is that $\mathscr{F}$ incorporates L in such a way that $L=0 \Rightarrow \mathscr{F}=0$. In that case every solution of $L=0$ yields a solution of the original nonlinear equation.

A second question is, assuming the pattern above is found to occur in practice, can it be used to find the transformation T? At pesent it is not clear if and when this can be done. It turns out that the key issue here is that of establishing what variables appear in T. Some clues to this may be found from the variables appearing in the admissible operator for the nonlinear equation but this method fails completely at times.

In each of the following examples we shall introduce a nonlinear differential equation and obtain, or cite, an admissible LB operator of the kind $\eta(h;u)\,\partial/\partial u$. We shall then obtain a straightforward group-theoretical characterisation of the linearisability of the equation, essentially by tracing the 'transformation' described above, between it and the target linear equation. At the same time we shall suggest how the linearising transformation T might be discovered. The examples will illustrate the value of being able to treat non-local variables in considering the linearisation of differential equations from the group-theoretical standpoint.

In the discussion which follows, the only restriction imposed *a priori* is the assumption that any transformation between two differential equations does not involve the (x^i). Transformations which depend on the (x^i) are excluded because, for example, in mechanical systems they are in general not canonical and may destroy the Hamiltonian structure of the problem as originally posed.

4.1. The Thomas equation

Example (4.1). The differential equation

$$u_{xt}+\alpha u_t+\beta u_x+\gamma u_x u_t=0 \tag{4.4}$$

where α, β, and γ are constant parameters, arises in consideration of certain fluid–solid exchange processes (see Whitham 1974) and was linearised by Thomas (1944). In the absence of any foreknowledge of the linearisability of (4.4) one might begin by finding the generators of all LB symmetries of the form $\eta\,\partial/\partial u$, where η depends on no more than t, x and u. These are the Lie *point* symmetries and they are found by solving equation (2.8) for this restricted case (see Ames (1972) and Bluman and Cole

(1974) for the method). In this case (2.8) reduces to the set of linear partial differential equations ('determining equations')

$$\eta_{xt}+\alpha\eta_t+\beta\eta_x=0 \qquad \eta_{tu}+\gamma\eta_t=0 \qquad \eta_{xu}+\gamma\eta_x=0 \qquad \eta_{uu}+\gamma\eta_u=0 \tag{4.5}$$

which have a solution

$$\eta=\gamma^{-1}h(x,t)\,\mathrm{e}^{-\gamma u} \tag{4.6}$$

where $h(x,t)$ is any solution of the linear equation

$$v_{xt}+v_t+\beta v_x=0 \tag{4.7}$$

and so the required generator is $\gamma^{-1}h(x,t)\,\mathrm{e}^{-\gamma u}\,\partial/\partial u$. It turns out that the approach of Kumei and Bluman (1981) is applicable to this problem, but instead of following it we wish to use this example to demonstrate how one may proceed in the more general case and perhaps find the linearising 'transformation'.

We have previously stressed that in general a transformation T is expected to carry (4.4) into a secondary equation of (4.7) (it may even carry it into a consequence of the secondary equation). We have also pointed out that the secondary equation, $\mathscr{F}(v)=0$, may contain the linear operator L in such a way that $L(v)=0$ implies $\mathscr{F}(v)=0$; that is, any solution of $L(v)=0$ is also a solution of $\mathscr{F}(v)=0$. With this in mind we emphasise *transformations on solutions* initially and attempt to find a relation between solutions of (4.4) and (4.7). Therefore, if $g(x,t)$ denotes a solution of (4.4) we look for a relation

$$\bar{g}=\bar{g}[h(x,t)] \tag{4.8a}$$

$$\bar{h}=\bar{h}[g(x,t)]. \tag{4.8b}$$

The bar over g is to distinguish the functional dependence of g on h from that of g on x or t, and likewise for the bar over h. The notation here is meant to indicate general dependence on, for example, $(\ldots, D^{-1}h, h, h_x, \ldots)$, and in order to proceed to a solution, an ansatz must be made about this dependence. In general, this is a very difficult step. In the present example we note that the operator admissible by the nonlinear equation involves only h and u and this motivates us to begin by assuming that $\bar{h}$ depends on g only. We then use this in equation (4.7) to obtain $\gamma\bar{h}_g-\bar{h}_{gg}=0$, having used the fact that g satisfies (4.4). This equation is easily solved to obtain

$$h=b\,\mathrm{e}^{\gamma g} \qquad g=\gamma^{-1}\log(h/b) \tag{4.9a, b}$$

where b is an arbitrary constant.

Replacing g by u and h by v in (4.9) yields a corresponding transformation in these variables. It is found to be an invertible $1-1$ point transformation between the equations (4.4) and (4.7). Therefore, this has turned out to be a particularly simple example in that neither non-local variables, secondary equations nor integral/differential consequences have appeared. Nevertheless, it illustrates two important points. Firstly, the admissible operator provides a target equation for the linearisation, in this case the linear part of the original nonlinear equation. Secondly, a transformation on solutions led to a transformation between the differential equations. In this case the variables appearing in the admissible operator suggested that a point transformation should be tried initially. This approach in forming the ansatz for the transformation worked out well here, but it is not always fruïtful; see example (4.3).

Finally we wish to apply the transformation (4.9) (in the form involving u and v) to the admissible operator, $\gamma^{-1}h(x,t)\,\mathrm{e}^{-\gamma u}\,\partial/\partial u$, of the nonlinear equation. We simply note that $\partial/\partial u=(\partial v/\partial u)\,\partial/\partial v$, and then make a change of variables to obtain the operator $h(\partial/\partial v)$ admitted by the linear equation (4.7).

4.2. The Burgers equation

Example (4.2). The well known Burgers equation, a model for turbulence, may be written in the form

$$u_t+uu_x-u_{xx}=0. \tag{4.10}$$

It admits no Lie point transformation (Katkov 1965) depending on an arbitrary solution of a linear differential equation. However, a LB operator of this kind can be found as follows (the method to be used is exemplary of the general method for finding LB operators admitted by differential equations).

Assume $\hat{X}=\eta\,\partial/\partial u$ is admitted by (4.10) with the ansatz that $\eta=\eta(t,x,D_x^{-1}u,u)$ and no more. It will also be assumed that (4.10) is a differential equation on a set of solutions where u and all its derivatives vanish sufficiently rapidly as $x\to-\infty$ (for all time) to ensure that $\int_{-\infty}^{x}f(u)\,\mathrm{d}x'$ includes no contribution at the lower limit. In that way D_x^{-1} may be interpreted as $\int_{-\infty}^{x}\mathrm{d}x'$.

We use the notation $\bar{p}$ for $D_x^{-1}u$ (corresponding to the classical notation p for u_x). Then with

$$D_t=\partial/\partial t+D_x^{-1}u_t(\partial/\partial\bar{p})+u_t\,\partial/\partial u+\ldots \tag{4.11a}$$

$$D_x=\partial/\partial x+u\,\partial/\partial\bar{p}+u_x\,\partial/\partial u+u_{xx}\,\partial/\partial u_x+\ldots \tag{4.11b}$$

the relevant part of $\hat{X}$ as extended is found using the recurrence relations (2.2). Thus

$$\hat{X}_{\mathrm{rel}}=\eta(\partial/\partial u)+\zeta^{x}(\partial/\partial u_x)+\zeta^{t}(\partial/\partial u_t)+\zeta^{xx}(\partial/\partial u_{xx}) \tag{4.12}$$

where

$$\zeta^{x}=\eta_x+u\eta_{\bar{p}}+u_x\eta_u \qquad \zeta^{t}=\eta_t+(D_x^{-1}u_t)\eta_{\bar{p}}+u_t\eta_u$$

$$\zeta^{xx}=\eta_{xx}+2u\eta_{x\bar{p}}+uu_x\eta_{u\bar{p}}+2u_x\eta_{xu}+u_x\eta_{\bar{p}}+u^2\eta_{\bar{p}\bar{p}}+u_{xx}\eta_u+u_x^2\eta_{uu}.$$

The coordinates ζ are identified using superscripts here, to avoid any confusion with standard partial derivative notation. The form $\eta_{\bar{p}}$ denotes partial derivative with respect to the independent variable $D_x^{-1}u$.

Using (4.10) and (4.12) in the determining equation (2.19) yields

$$\eta_t-\tfrac{1}{2}u^2\eta_{\bar{p}}+u_x\eta+u\eta_x+u^2\eta_{\bar{p}}-\eta_{xx}-2u\eta_{x\bar{p}}$$
$$-2uu_x\eta_{u\bar{p}}-2u_x\eta_{xu}-u^2\eta_{\bar{p}\bar{p}}-u_x^2\eta_{uu}=0 \tag{4.13}$$

where application of $\omega(u)\doteq0$ made use of the integral consequence

$$D_x^{-1}u_t+\tfrac{1}{2}u^2-u_x=0 \tag{4.14}$$

as well as equation (4.10) itself. Treating the left-hand side of (4.13) as a polynomial in powers of u_x yields the following equations for η:

$$\eta_t+\tfrac{1}{2}u^2\eta_{\bar{p}}+u_x\eta+u\eta_x-\eta_{xx}-2u\eta_{x\bar{p}}-u^2\eta_{\bar{p}\bar{p}}=0 \tag{4.15a}$$

$$\eta-2\eta_{xu}-2u\eta_{u\bar{p}}=0 \tag{4.15b}$$

$$\eta_{uu}=0. \tag{4.15c}$$

By using the last two equations of this set

$$\eta = 2(\psi_x(t, x, \bar{p}) + u\psi_{\bar{p}}(t, x, \bar{p}))$$

and if ψ is assumed to be of separable form $h(t, x)\varphi(\bar{p})$, (4.15a) can be used to obtain the solution

$$\eta = (hu + 2h_x)\,\mathrm{e}^{\bar{p}/2} \tag{4.16}$$

where $h(t, x)$ is any solution of

$$v_t - v_{xx} = 0, \tag{4.17}$$

the linear heat equation.

The admissible operator $\eta\ \partial/\partial u$, with η given by (4.16), provides the target equation of the linearisation (4.17) and also suggests that perhaps a relationship on solutions should be sought in the form $\bar{h} = \bar{h}(g, D_x^{-1}g)$, where g is a solution of the nonlinear equation (4.10). Inserting this into (4.17) and using (4.10) and (4.14) to eliminate the time derivatives of g yields the following power series in g_x:

$$\bar{h}_{gg}g_x^2 - (\bar{h}_g g + 2h_{gD_x^{-1}g})g_x - \bar{h}_{(D_x^{-1}g)^2}g^2 + \tfrac{1}{2}\bar{h}_{D_x^{-1}g} = 0.$$

This in turn yields a set of partial differential equations for $\bar{h}(g, D_x^{-1}g)$:

$$\bar{h}_{gg} = 0 \qquad \bar{h}_g g + 2\bar{h}_{gD_x^{-1}g} = 0 \qquad \bar{h}_{(D_x^{-1}g)^2}g^2 + \tfrac{1}{2}\bar{h}_{D_x^{-1}g} = 0. \tag{4.18}$$

The set (4.18) is easily solved and two solutions will now be considered separately.

4.2.1. Linearisation by the Cole–Hopf transformation. The first solution is

$$h = \exp(-\tfrac{1}{2}D_x^{-1}g) \qquad g = -2(h_x/h) \tag{4.19a, b}$$

and corresponding to this transformation on solutions we have

$$v = \exp(-\tfrac{1}{2}D_x^{-1}u) \qquad u = -2(v_x/v) \tag{4.20a, b}$$

acting on the equations. The latter is the well known Cole–Hopf transformation which transforms the Burgers equation into

$$2v^{-2}(v_x - vD_x)(v_t - v_{xx}) = 0 \tag{4.21}$$

and *vice versa*. The last equation may be rewritten in the form

$$2D_x((v_t - v_{xx})v^{-1}) = 0 \tag{4.21}$$

which is a first differential consequence of a secondary equation of the linear heat equation. It admits the operator $h(\partial/\partial v)$ admitted by the linear heat equation, as may easily be verified.

Equation (4.21) incorporates the quantity $v_t - v_{xx}$ in such a way that every solution of the latter is a solution of (4.21) also, and so by the transformation T, (4.19), a corresponding solution of the nonlinear equation (4.10) can be found. The reverse is not quite true however. By reviewing the way (4.19) was obtained the reader will see that a solution, g, of the nonlinear equation (4.14) must also satisfy equation (4.14) before it will yield a solution of the heat equation through the Cole–Hopf transformation. On the restricted solution set we have chosen, this condition is, of course, satisfied. Moreover, the transformation is 1–1, because even though (4.19b) allows for many solutions h to yield the same g, (4.19a) rectifies this situation. Therefore, it can be said that on a set of solutions which vanishes sufficiently rapidly as $x \to -\infty$, the Cole–Hopf transformation is a 1–1 linearising map for the Burgers equation.

There is nothing new in the results of the last paragraph, which follow directly from the Cole–Hopf transformation. Our point is that the underlying group structure is associated with a true linearisation, albeit on a restricted solution set. We complete our consideration of the group-theoretical aspect by showing how to transform the operator $h(\partial/\partial v)$ admitted by the heat equation and the secondary equation (4.21) into $\eta(h; u)\,\partial/\partial u$ admitted by the Burgers equation, where η is given by (4.16). We apply (4.20) to $h(\partial/\partial v)$ by using equation (3.2). Part of the extended operator is

$$\hat{Y}(v) = \ldots + D_x^{-1}h(\partial/\partial(D_x^{-1}v)) + h(\partial/\partial v) + h_x(\partial/\partial v_x) + \ldots.$$

The idea is to pick up all of the terms involving $\partial/\partial u$ and thereby obtain the defining term in $\hat{X}(u)$. Since u is a function of v and v_x only these are

$$\frac{\partial}{\partial v} = \ldots + \frac{\partial u}{\partial v}\frac{\partial}{\partial u} + \ldots = \ldots + 2\frac{v_x}{v^2}\frac{\partial}{\partial u} + \ldots$$

and

$$\frac{\partial}{\partial v_x} = \ldots + \frac{\partial u}{\partial v_x}\frac{\partial}{\partial u} + \ldots = \ldots - \frac{2}{v}\frac{\partial}{\partial u} + \ldots.$$

Equation (4.20) must now be used to obtain v and v_x in terms of the u variables giving

$$\hat{X}(u) = \ldots - (hu + 2h_x)\,e^{\bar{v}/2}(\partial/\partial u) + \ldots, \tag{4.22}$$

admitted by the Burgers equation. We have, therefore, traced the admissible group from the linear equation, through the secondary equation to the nonlinear equation. The reverse procedure might just as well have been followed.

The Cole–Hopf transformation (4.20) has been criticised because when u vanishes as $x \to -\infty$, v remains finite and so the solutions, h, are functionally very different from the solutions, g. This difficulty is removed by using the second solution of (4.18), now to be discussed.

4.2.2. Linearisation by the transformation of Taflin. A second solution to (4.18) is

$$\bar{h} = -\tfrac{1}{2}g\exp(-\tfrac{1}{2}D_x^{-1}g) \qquad \bar{g} = -2h(1 + D_x^{-1}h)^{-1} \tag{4.23a, b}$$

with the corresponding transformation in the u and v variables:

$$v = -\tfrac{1}{2}u\exp(-\tfrac{1}{2}D_x^{-1}u) \qquad u = -2v(1 + D_x^{-1}v)^{-1}. \tag{4.24a, b}$$

This transformation is due to Taflin (1981) who showed that it has certain desirable properties on the Schwartz space of functions decreasing rapidly at infinity; for example when u is the Schwartz space, so is v and *vice versa*. For this reason we choose to discuss this transformation in the context of solutions of the Burgers equation which are in the Schwartz space, a condition somewhat more restrictive than that introduced at the beginning of our discussion of the Burgers equation. Our choice then implies that we also restrict ourselves to solutions of the linear heat equation which are in the Schwartz space.

The transformation (4.24) transforms the nonlinear Burgers equation into

$$2D_x\left(\frac{D_x^{-1}(v_t - v_{xx})}{1 + D_x^{-1}v}\right) = 0 \tag{4.25}$$

and conversely. Equation (4.25) is the *first differential consequence* of a secondary equation of the *first integral consequence* of the heat equation. It admits the operators $h(\partial/\partial v)$, $h_x(\partial/\partial v)$, etc, admitted by the latter.

The way in which the quantity $v_t - v_{xx}$ appears in (4.25) requires that a solution, $h(x, t)$, of the heat equation must also satisfy the equation

$$D_x^{-1}v_t - v_x = 0 \tag{4.26}$$

in order to satisfy equation (4.25), but of course (4.26) is satisfied by each h in the Schwartz space. Each such solution then yields a solution of the Burgers equation through the transformation T of (4.23). If we begin with a solution, g, of the Burgers equation, it must also satisfy equation (4.14) to yield a solution of the heat equation, but this condition is also met when g is in the Schwartz space. Add to all this the fact that (Taflin 1981) (4.23) is 1–1 on the Schwartz space and we have the result that it is a 1–1 linearising map for the Burgers equation on that space. Once more our point is not that these are new results, but that the underlying group structure is associated with a true linearisation, albeit on a restricted solution set.

We complete our discussion of the group-theoretical aspect by showing that Taflin's transformation (4.24) acts on the operator $h_x(\partial/\partial v)$, admitted by the heat equation and by (4.25), to yield the operator (4.22) admitted by the Burgers equation; this is an example of the application of equation (3.2) involving the manipulation of non-local variables. Firstly we note that part of the extended operator is

$$Y(v) = \ldots + h(\partial/\partial(D^{-1}v)) + h_x(\partial/\partial v) + h_{xx}(\partial/\partial v_x) + \ldots.$$

Now u is a function of v and $D^{-1}v$ only and so we pick up the defining term of the transformed operator by computing

$$\frac{\partial}{\partial v} = \ldots + \frac{\partial u}{\partial v}\frac{\partial}{\partial u} + \ldots = \ldots - \frac{2}{1+D_x^{-1}v}\frac{\partial}{\partial u} + \ldots$$

and

$$\frac{\partial}{\partial(D_x^{-1}v)} = \ldots + \frac{\partial u}{\partial(D_x^{-1}v)}\frac{\partial}{\partial u} + \ldots = \ldots + \frac{2v}{(1+D_x^{-1}v)^2}\frac{\partial}{\partial u} + \ldots.$$

Equation (4.24a) is now used to obtain v and $D_x^{-1}v$ in terms of the u variables giving precisely (4.22) as claimed.

4.3. *Linearisation of an ordinary differential equation of classical mechanics*

Example (4.3). The ordinary differential equation

$$u_{xx} + \omega^2 u + Ku^{-3} = 0 \tag{4.27}$$

where ω is an arbitrary function of x (the time), arises in consideration of the motion of a charged particle in a uniform, time-dependent, axial magnetic field (Lewis 1967, Reid 1974). In that case K is the square of the angular momentum. The equation appears frequently in the literature beginning perhaps with Ermakov (1880) and it can be linearised in two different ways, both of which will now be considered.

4.3.1. Linearisation to a third-order equation. Equation (4.27) admits the LB operator

$$(\tfrac{1}{2}h_x u - hu_x)\,\partial/\partial u \tag{4.28}$$

where h is any solution of the third-order linear equation

$$v_{xxx}+4\omega^2 v_x+4\omega\omega_{xx}v=0. \tag{4.29}$$

This is the equivalent form, by equation (2.5), of the admissible operator appearing in Ray and Reid (1979). It is noteworthy that the target equation (4.29) is not the linear part of (4.27).

Although no point transformation, with u a function of v only, can transform (4.27) into (4.29), such a transformation might very well transform it into the integral consequence of some secondary equation of (4.29). So we make the ansatz that $\bar{h}=\bar{h}(g)$ solves (4.29), where g is a solution of (4.27), and substitute in (4.29) to get

$$\bar{h}_{ggg}g_x^2-(3\bar{h}_{gg}g-3\bar{h}_g)g_x\omega^2-3K(\bar{h}_{gg}g^{-3}-\bar{h}_g g^{-4})g_x-(2\bar{h}_g g-4\bar{h})\omega\omega_x=0$$

having used equation (4.27) to substitute for g_{xx} and g_{xxx}. The above equation has the solution

$$h=g^2 \qquad g=h^{1/2}. \tag{4.30a, b}$$

Applying (4.30b), in the form $u=v^{1/2}$, to equation (4.27) yields (Ray and Reid 1979)

$$v_x^2-2vv_{xx}-4\omega^2v^2-4K=0. \tag{4.31}$$

This is the same equation (4.2) discussed earlier and so we recognise it as a first integral of (4.29). Because of the integrating factor v involved, it is, in the language we have been using, a first integral consequence of the secondary equation

$$v(v_{xxx}+4\omega^2v_x+4\omega\omega_x v)=0 \tag{4.32}$$

of (4.29).

Since K is a constant (some given number), equation (4.31) is only one of many first integral consequences of (4.32) corresponding to the many possible different values of K, and so its solution set is smaller than that of (4.32) or (4.29). In fact we can be more explicit, for the general solution of (4.29) is known (Chini 1897) and is given by

$$v=Az_1^2+Bz_1z_2+Cz_2^2 \tag{4.33}$$

where z_1 and z_2 are linearly independent solutions of the linear harmonic oscillator equation $z_{xx}+\omega^2z=0$. When (4.33) is substituted in (4.31) it is found to satisfy it on condition that

$$(C^2-4AB)W^2=4K \tag{4.34}$$

where W is the Wronskian of z_1 and z_2. Employing (4.30b) we may now write the general solution of the original nonlinear equation as $u=(v^*)^{1/2}$, where v^* is the solution of the linear equation (4.29) as restricted by (4.34). Once again we have been able to obtain a one-to-one linearising map, in this case by picking out the appropriate subset of the general solution of the linear equation.

To see the underlying group-theoretical structure associated with this linearisation we note that when the transformation T of (4.30) (written in terms of u and v) is applied to the free operator (4.28), it yields the free operator $(h_xv-hv_x)\,\partial/\partial v$, admitted by equations (4.29) and (4.31). The actual calculations here are simple and the details are omitted.

It is remarkable that the linearisation of equation (4.27) has been characterised group theoretically in terms of the operator $(h_x v - hv_x)\,\partial/\partial v$ and not in terms of the simpler operator $h(\partial/\partial v)$, also admitted by the linear equation (4.29). This situation is related to the previously mentioned fact that (4.31) is only one of many integral consequences of (4.29) and that its solution set is smaller than that of (4.29). Recall that when a differential equation admits a Lie point symmetry operator it is equivalent to its solution manifold being transformed into itself under the corresponding Lie point group of transformations. If we integrate the differential equation to produce a new equation with a smaller solution set we can expect that some or all of the original symmetries will be lost. This is just what happens here, for the operator $h(\partial/\partial v)$ is not admitted by (4.31) and so its counterpart $(h/u)\,\partial/\partial u$ is not admitted in the usual sense by the nonlinear equation (4.27). However, when the constant K is interpreted as the variable initial value of $u^3 u_{xx} + \omega^2 u^4$ it can be shown that $(h/u)\,\partial/\partial u$ is admitted by equation (4.31) (see Cullen 1982 for details).

4.3.2. Linearisation to the linear part. It is suggested by the foregoing that the condition that $h(x)$ satisfy the third-order equation (4.29) might be replaced by, say, $k(x)$ satisfying the equation

$$\tilde{v}_{xx} + \omega^2 \tilde{v} + K\tilde{v}^{-3} = 0 \tag{4.35}$$

so long as $h = k^2$, and subject to the restriction implied by (4.34). Since K in (4.35) may assume the value zero we may further expect that the operator

$$(kk_x u - k^2 u_x)\,\partial/\partial u \tag{4.36}$$

(obtained from (4.28)) is admitted by equation (4.27), where k satisfies the linear equation

$$\tilde{v}_{xx} + \omega^2 \tilde{v} = 0. \tag{4.37}$$

This is, indeed, found to be the case and it raises the possibility that the original nonlinear equation (4.27) can be transformed to its linear part, equation (4.37), or to one of its consequences.

To find this transformation, make the ansatz that $\bar{k} = \bar{k}(g, D_x^{-1} g^{-2})$, where g is a solution of (4.27). The motivation for introducing the variable $D_x^{-1} g^{-2}$ is admittedly weak, but it does appear repeatedly in work dealing with equations like (4.27)—see, for example, Eliezer (1979) and Leach (1981). Substituting this ansatz into equation (4.37) and proceeding in the usual manner yields a set of partial differential equations for $\bar{k}$ which can be solved to obtain

$$k = g\,\exp(-\sqrt{K} D_x^{-1} g^{-2}) \qquad g = (4K)^{1/4} k (D_x^{-1} k^{-2})^{1/2}. \tag{4.38a, b}$$

In terms of u and $\tilde{v}$ the transformation takes the form

$$\tilde{v} = u\,\exp(-\sqrt{K} D_x^{-1} u^{-2}) \qquad u = (4K)^{1/4} \tilde{v} (D_x^{-1} \tilde{v}^{-2})^{1/2} \tag{4.39a, b}$$

and when u is substituted into (4.27) it yields the secondary equation of (4.37):

$$(4K)^{1/4} (\tilde{v}_{xx} + \omega^2 \tilde{v})(D_x^{-1} \tilde{v}^{-2})^{1/2} = 0. \tag{4.40}$$

This result represents an entirely satisfactory linearisation of the original equation (4.27) because the general solution of the linear equation (4.37) satisfies (4.40) and so is transformed into a solution of (4.27) via (4.38*b*). Since the latter solution contains two arbitrary constants it is the general solution of (4.27).

Turning now to the group-theoretical aspect we find that equation (4.40) admits the operator

$$\hat{Y} = (kk_x\tilde{v} - k^2\tilde{v}_x)\,\partial/\partial\tilde{v} \tag{4.41}$$

which is obtained when the transformation (4.39) is applied to the operator (4.36). It has the same form as (4.36) and it is an interesting exercise in the manipulation of non-local variables to obtain it and to show explicitly that it is admitted by (4.40). We begin by noting that the operator (4.36) as extended is

$$\hat{X} = \ldots + \eta_f(\partial/\partial(D_x^{-1}u^{-2})) + \ldots + \eta(\partial/\partial u) + \ldots. \tag{4.42}$$

The two terms explicitly shown are the relevant ones because $\tilde{v}$ is a function of the independent variables u and $D_x^{-1}u^{-2}$ and so the coordinate of $\partial/\partial\tilde{v}$ in the transformed operator may be obtained from them with

$$\partial/\partial u = \ldots + (\partial\tilde{v}/\partial u)\,\partial/\partial\tilde{v} + \ldots$$

and

$$\frac{\partial}{\partial(D_x^{-1}u^{-2})} = \ldots + \frac{\partial\tilde{v}}{\partial(D_x^{-1}u^{-2})}\frac{\partial}{\partial\tilde{v}} + \ldots.$$

These two are easily evaluated using the transformation (4.39) and they are

$$\partial/\partial u = \ldots + (4K)^{-1/4}(D_x^{-1}\tilde{v}^{-2})^{1/2}\,\partial/\partial\tilde{v} + \ldots$$

$$\partial/\partial(D_x^{-1}u^{-2}) = \ldots + \sqrt{K}\tilde{v}(\partial/\partial\tilde{v}) + \ldots.$$

The coordinate η_f is the first-order increment in $D_x^{-1}u^{-2}$ and is found to be $-2D_x^{-1}(\eta u^{-3})$ which integrates to $-k^2u^{-2}$. Finally, transforming to the $\tilde{v}$ variables,

$$\eta_f = \tfrac{1}{2}k^2\tilde{v}^{-1}(D_x^{-1}\tilde{v}^{-2})^{-1}.$$

Substituting for η, η_f, $\partial/\partial u$ and $\partial/\partial(D_x^{-1}u^{-2})$ in (4.42) then yields the operator (4.41).

Dispensing with the constant $(4K)^{1/4}$, the first differential consequence of (4.40) is

$$D_x(\tilde{v}_{xx} + \omega^2\tilde{v})(D_x^{-1}\tilde{v}^{-2})^{1/2} + \tfrac{1}{2}(\tilde{v}_{xx} + \omega^2\tilde{v})(D_x^{-1}\tilde{v}^{-2})^{-1/2}\tilde{v}^{-2} = 0 \tag{4.43}$$

and the relevant part of the operator (4.41) as extended is

$$\hat{Y}_{\text{rel}} = -k^2\tilde{v}^{-2}\,\partial/\partial(D_x^{-1}\tilde{v}^{-2}) + \hat{Y}.$$

Applying this to (4.40) gives

$$[\hat{Y}(\tilde{v}_{xx} + \omega^2\tilde{v})](D_x^{-1}\tilde{v}^{-2})^{1/2} - \tfrac{1}{2}k^2(\tilde{v}_{xx} + \omega^2\tilde{v})(D_x^{-1}\tilde{v}^{-2})^{-1/2}\tilde{v}^{-2}$$

and this is zero on the manifold of (4.40) and (4.43), which proves that the operator (4.41) is admitted by the secondary equation (4.40). It is easy to verify that it is also admitted by (4.27) itself.

Once again it is remarkable that the linearisation of equation (4.27) has been characterised group theoretically by means of an admissible operator of the linear equation and which is not of the simple form $k(\partial/\partial\tilde{v})$. However, this latter operator is admitted by the secondary equation (4.40) and moreover $k(\partial/\partial\tilde{v})$ can be transformed into the following operator admitted by the nonlinear equation (4.27)

$$[k\exp(\sqrt{K}D_x^{-1}u^{-2}) - (4K)^{1/4}D_x^{-1}(ku^{-3}\exp(3\sqrt{K}D_x^{-1}u^{-2}))$$
$$\times u\exp(-2\sqrt{K}D_x^{-1}u^{-2})](\partial/\partial u).$$

For the details of the related calculations see Cullen (1982).

5. Summary and conclusions

The recent augmentation of the theory of LB groups by Konopelchenko and Mokhnachev (1979, 1980) introducing non-local variables into the formalism is an important development. In the present work we have shown that the LB operators of these authors are related to the LB operators of Ibragimov and Anderson (1977) by a change of variables. We have observed that if a given differential, etc, equation admits a LB operator, then many related ('secondary') equations admit the same operator. These secondary equations are in addition to the integrodifferential consequences of the original equation.

The Burgers equation is shown to admit a LB free group operator which involves a non-local variable. This is essentially the same operator discussed by Ibragimov (1980), without recourse to the non-local formalism, but at the expense of introducing an extra differential equation, the Burgers potential equation. The Burgers equation is known to be linearisable by the Cole–Hopf transformation and by a more recent transformation due to Taflin. By employing the above LB free group operator (which depends on an arbitrary solution of the linear heat equation) and the concept of secondary equations, a straightforward group-theoretical characterisation of these linearisations is found. The ordinary differential equation, $u_{xx}+\omega^2u+Ku^{-3}=0$, can be linearised in two different ways, each of which can be characterised group theoretically by using the concepts of admissible free group operators, secondary equations, and integral consequences.

In all the above work the non-local variables were treated as independent, on the same footing as the usual independent variables of LB theory, without giving rise to any contradictions or other difficulties.

In the course of our study of linearisation we considered certain 'transformations' between nonlinear and linear equations. We wish to emphasise that these 'transformations' in general occur in two stages; firstly a transformation T to a secondary equation and then a relationship such as those of § 2.3 to the linear equation itself. The secondary equation is often of such a form that it reduces to the linear equation on some particular function space or set of solutions of the linear equation.

In pursuing the kind of group-theoretical approach to linearisation described here, the most difficult step will always be the search for an admissible free group operator of the nonlinear equation. The difficulty arises mainly from the unlimited number of independent variables which may appear in LB group operators. Once such an admissible group is found for some nonlinear equation, it is the authors' conjecture that a linearisation exists and can be found. Granted, a second ansatz must be made about the independent variables appearing in the linearising transformation; but with linearisability suggested by, and a target equation provided by an admissible free group, a determined effort will almost certainly reveal the needed transformation.

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